Boltzmann eq°

$$\frac{\partial f(\vec{q},\vec{p},t)}{\partial t} + \{f, H_1\} = \int_{\vec{p}_2,\theta,\varphi} d^3\vec{p}_2\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1 - \vec{v}_2|\, (f_1' f_2' - f_1 f_2)$$

where $f_i^{(\prime)} = f(\vec{q}, \vec{p}_i^{(\prime)})$

(i) closed equation for f

(ii) irreversible

Boltzmann H theorem.

Theorem: $\mathcal{H}(t) = \int d\vec{p}\, d\vec{q}\, f(\vec{q},\vec{p},t) \ln f(\vec{q},\vec{p},t)$ is a decreasing function of time.

$$\frac{d\mathcal{H}}{dt} = \int d\vec{p}_1\, d\vec{q}\, \ln f_1\, \partial_t f_1$$

$$= \int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1 - \vec{v}_2|\, \ln f_1\, (f_1' f_2' - f_1 f_2) + 0$$

$\vec{p}_1 \leftrightarrow \vec{p}_2$

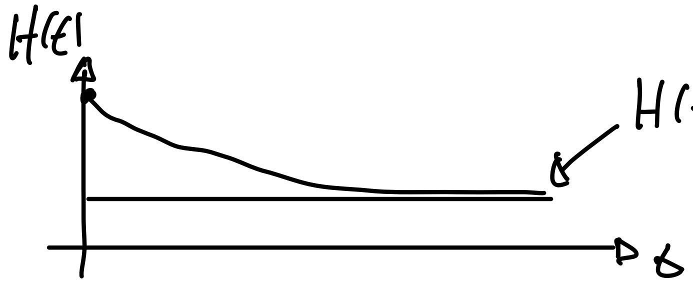
$$= \frac{1}{2} \int d\vec{q}\, d\vec{p}_1\, d\vec{p}_2\, d^2\sigma\, |\vec{v}_1 - \vec{v}_2|\, \underbrace{(f_1' f_2' - f_1 f_2)}_{\text{antisymmetric in } \vec{p}_i \leftrightarrow \vec{p}_i{}'} (\ln f_1 + \ln f_2)$$

$\vec{p}_i \leftrightarrow \vec{p}_i{}'$

$$= \frac{1}{4} \int d\vec{q}\, d\vec{p}_1\, d\vec{p}_2\, d^2\sigma\, |\vec{v}_1 - \vec{v}_2|\, \underbrace{(f_1' f_2' - f_1 f_2)}_{\text{this term}} \underbrace{(\ln f_1 f_2 - \ln f_1' f_2')}_{\text{and that term have}}$$

opposite signs

$$\Rightarrow \frac{d}{dt} H(t) \leq 0 \quad \& \quad \frac{d}{dt} H(t) = 0 \text{ if } f_1' f_2' = f_1 f_2$$



H(t) is monotonously decreasing.

Q: What is the steady state?

(i) From Pset 2: minimizing H(t) under constraint leads to equilibrium physics

(ii) Is this the only solution?

$$\frac{d}{dt}H=0 \Rightarrow f_1' f_2' = f_1 f_2 \Leftrightarrow \ln f_1(\vec{q}, \vec{p}_1{}') + \ln f_1(\vec{q}, \vec{p}_2{}') = \ln f_1(\vec{q}, \vec{p}_1) + \ln f_1(\vec{q}, \vec{p}_2)$$

$$\Leftrightarrow \ln f(\vec{q}, \vec{p}, t) \text{ is a collisional invariant}$$

They are only 5 such quantities:

1. particle number $2 = 1+1 \rightarrow 1+1=2$
2. momentum components $p_{1,\alpha} + p_{2,\alpha} = p_{1,\alpha}' + p_{2,\alpha}'$
3. (kinetic) energy $\vec{p}_1{}^2 + \vec{p}_2{}^2 = \vec{p}_1{}'^2 + \vec{p}_2{}'^2$

In each box, $\ln f$ is a linear combination of these with $\vec{q}$-varying prefactors

$$\Rightarrow \ln f_1(\vec{p}, \vec{q}) = \tilde{\gamma}(\vec{q}) - \vec{\alpha}(\vec{q}) \cdot \vec{p} - \beta(\vec{q}) \frac{\vec{p}^2}{2m}$$

Local equilibrium

Using $\tilde{\gamma} = -(\ln \gamma(\vec{q}))\beta(\vec{q})\mu(\vec{q})$ leads to

$$f_1^{LEQ}(\vec{q}, \vec{p}) = \gamma(\vec{q})\, e^{-\vec{\alpha}(\vec{q}) \cdot \vec{p} - \beta(\vec{q})\left[\frac{\vec{p}^2}{2m} + \mu(\vec{q})\right]}$$

$\forall\, \gamma(\vec q), \vec\alpha(\vec q), \beta(\vec q)$, f_1^{LEQ} is such that $\frac{d}{dt} H[f_1^{LEQ}] = 0$

$\Rightarrow$ lots of fixed points of $H(f_1(\vec q,\vec p,t))$

Q: Is f^{LEQ} a solution of Boltzmann equation? Not in general

$$\text{Only if } -\{f^{LEQ},H_1\} + \underbrace{\int d^3\vec p_2\, d^2\sigma\, |\vec v_1-\vec v_2|\,(f_1'f_2'-f_1f_2)}_{\equiv C(f,f)} = 0$$

* If $f=f^{LEQ}$, then $C(f,f)=0 \Rightarrow$ invariant by collision
* $\{f^{LEQ},H_1\} \overset{?}{=} 0 \to$ Not in general
  $\to$ Yes if γ & β are constants & $\vec\alpha=0$

Global equilibrium

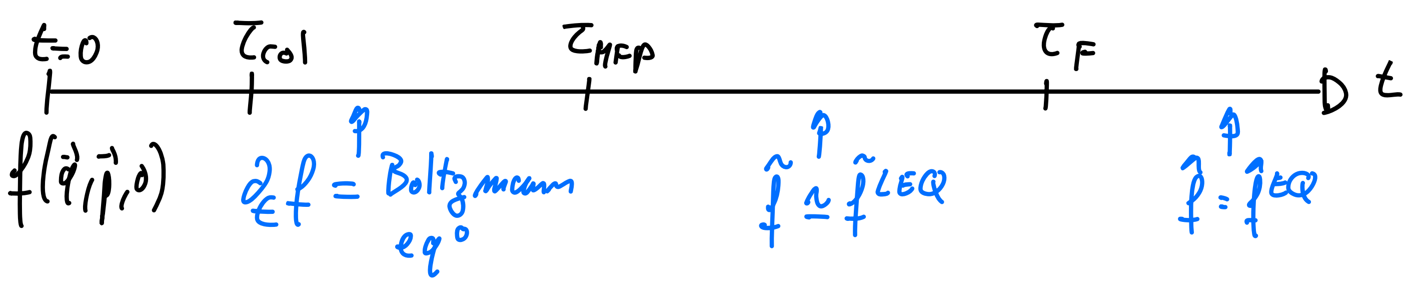
If $\vec\alpha=0$ & γ,β are constants, $f=f^{EQ}=\gamma e^{-\beta H_1(\vec q)}$

Since $f^{LEQ}(\vec q,\vec p) = g(H_1(\vec q,\vec p))$, we have $\{f^{LEQ},H_1\} = g'\times\{H_1,H_1\}=0$

Otherwise $\{f^{LEQ},H_1\}\neq 0 \Rightarrow$ not a steady state solution.

Dynamics

$$\partial_t f = -\underbrace{\{f,H_1\}}_{\text{relaxes on time scale } \tau_F} + \underbrace{C(f,f)}_{\text{relaxes on a time scale } \sim \tau_{MFP}}$$



Idea:

(i) Collisions make $\hat{f}$ equilibrate locally so that $\hat{f} \simeq \hat{f}^{LEQ}$ for $t \gg t_{MFP}$

(ii) Then, there is a slower evolution to $\hat{f}^{EQ}$ $\Rightarrow$ Q: how?

Canonical distribution: f^{EQ} is the canonical distribution even though the system is isolated $\Rightarrow$ contradiction?

No! $\hat{f}$ is the one-body distribution & single particles are NOT isolated $\Rightarrow$ they exchange energy during collisions. The other $N-1$ particles act as a thermostat.

Why is f^{EQ} independent from $V(\vec{q}_i - \vec{q}_j)$?

$$H = \underbrace{\sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + U(q_i)}_{\mathcal{O}(1)} + \underbrace{\frac{1}{2}\sum_{j\neq i} V(\vec{q}_i - \vec{q}_j)}_{\mathcal{O}(n d^3 \ll 1)} \simeq \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_i)$$

$$P^{EQ}(\{\vec{q}_i, \vec{p}_i\}) \propto \prod_i e^{-\beta\left[\frac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_i)\right]} \Rightarrow \text{Ideal gas}$$

$\rightarrow$ enough collisions to equilibrate

$\rightarrow$ not enough to alter the statistics

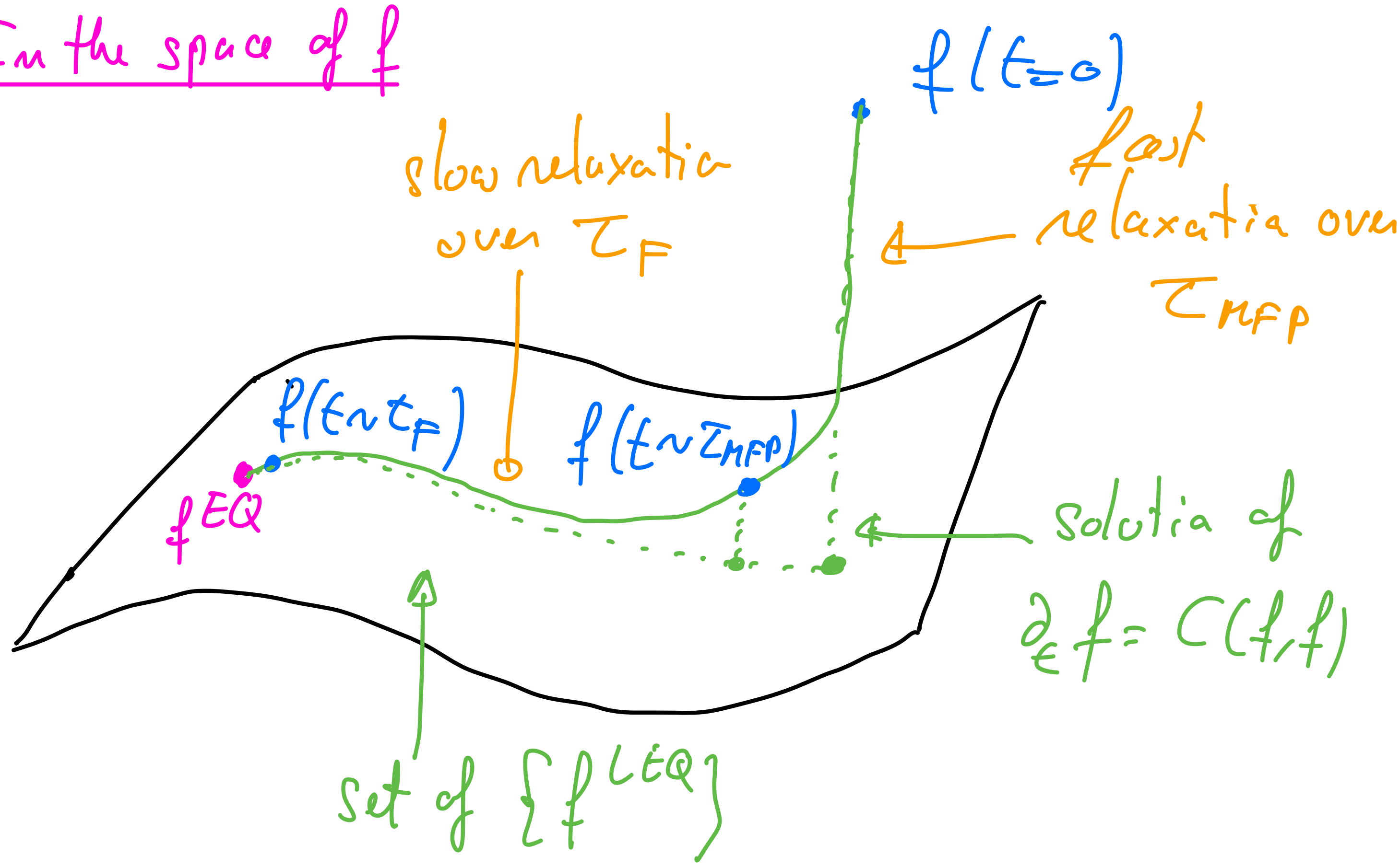
2.4) Transport properties & hydrodynamic equations

2.4.1) Relaxation of the phase space density

Let us characterise the relaxation of f to f^{EQ}.
To focus on bulk transport properties, we set $U=0$ so that

$$\partial_t f = -\frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial \vec{q}} + C(f,f)$$

In the space of f



We have argued that the only slow modes of the system correspond to conserved quantities: density $n(\vec{r},t)$, momentum $\vec{\pi}(\vec{r},t)$ & energy $\varepsilon(\vec{r},t)$ fields so that we can express all other slow quantities and their evolutions in terms of n, $\vec{\pi}$ & ε. Let's see how this will turn out to be true for f.

Time scale separation & perturbation theory

We rewrite the (BE) as $\partial_t f = -\frac{1}{\tau_F} L_F f + \frac{1}{\tau_{MFP}} \hat{C}(f,f)$ (Δ)

where $L_F f = \tau_F \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} f \underset{\tau_F \to 0}{\sim} \mathcal{O}(1)$

and $\hat{C}(f,f) = \tau_{MFP}\, C(f,f) \underset{\tau_{MFP} \to 0}{\sim} \mathcal{O}(1)$

⇒ We have made the scaling of the various terms explicit.

Next, we want to characterize the relaxation of f over $t \propto \tau_F$

$\Rightarrow t = \hat{t}\, \tau_F$, with $\hat{t} \sim \mathcal{O}(1)$ & $\frac{\partial}{\partial t} = \frac{1}{\tau_F} \frac{\partial}{\partial \hat{t}}$

(Δ) × τ_F ⇒ $\boxed{\partial_{\hat{t}} f = -L_F f + \frac{1}{\varepsilon} \hat{C}(f,f) \ (*)}$ with $\varepsilon = \frac{\tau_{MFP}}{\tau_F} \ll 1$

Perturbation theory

$$f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon f_1(\vec{q},\vec{p},t) + \mathcal{O}(\varepsilon^2)$$

$(*) \Rightarrow \partial_{\hat{t}} f_0 + \varepsilon \partial_{\hat{t}} f_1 = -L_F f_0 - \varepsilon L_F f_1 + \frac{1}{\varepsilon} \hat{C}(f_0,f_0) + \hat{C}(f_0,f_1) + \hat{C}(f_1,f_0)$

Order by order

$\mathcal{O}(\frac{1}{\varepsilon})$: $0 = C(f_0,f_0) \Rightarrow f_0 = f^{LEQ}$

$\mathcal{O}(1)$: $\partial_{\tilde{t}} f_0 + L_1 f_0 = C(f_0, f_1) + C(f_1, f_0)$

$\mathcal{O}(\varepsilon)$: would require f_2.

leading order & slow fields

$$f_0(\vec{q}, \vec{p}, t) = \tilde{\gamma}(\vec{q}, t)\, e^{-\vec{\alpha}(\vec{q}, t)\cdot\vec{p} - \beta(\vec{q}, t)\left[\frac{\vec{p}^2}{2m} + U(\vec{q})\right]}$$